Finding limits:

Consider
$$f(x) = \begin{cases} x^2 & x < -1 \\ 2 & 0 \le x \le 2 \\ x-3 & x > 2 \end{cases}$$

(piecewise function)

(a) $\lim_{x \to -1^-} f(x) = (-1)^2 = 1$

(b) $\lim_{x \to 0} f(x) = 2$

(c) $\lim_{x \to 2} f(x) = \text{DNE}$

$$\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= 2 \\ \lim_{x \to 2^+} f(x) &= -1 \end{aligned}\right\}$$

In general, to find the $\lim_{x \to a} f(x)$, we follow these steps:

* Check $f(x)$ is not a piecewise function

* Assume $f(x)$ is not a piecewise function (otherwise see the previous example)

1- plug "a" into $f(x)$!

Case, if $f(a)$ exists, (not undefined!) done!

$\Longrightarrow$ f is continuous at an open neighborhood of "a"

Eg: $\lim_{x \to 2} (x^2 - 4) = (2)^2 - 4 = \boxed{0}$

not a piecewise.

Eg: $\lim_{x \to 2} \left(\frac{x+1}{x-1}\right) = \frac{2+1}{2-1} = \frac{3}{1} = \boxed{3}$

Case 2: $f(a) =$ undefined (or error)

If $\lim_{x \to a^{\pm}} f(x) = \pm\infty$ (unbounded value!)

then $\boxed{x = a}$ is called a vertical asymptote

Eg: $\lim_{x \to 0} \left(\frac{1}{x}\right) \longrightarrow \frac{1}{0}$ undefined! (or f is not defined at "0")

$\lim_{x \to 0^-} \left(\frac{1}{x}\right) = -\infty$ (check with $x = -10^{-10}$)

$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = +\infty$ (check with $x = +10^{-10}$)

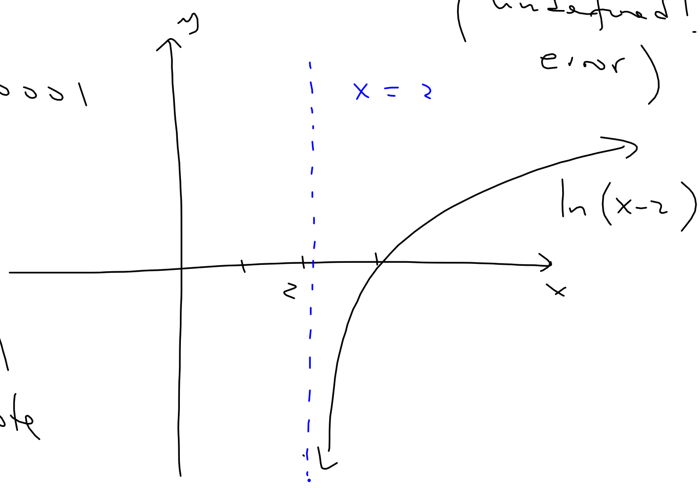
$\boxed{x = 0}$ is a vertical asymptote!

Ex: $\lim_{x \to 2^+} \ln(x-2) \longrightarrow \ln(2-2) = \ln(0)$ (undefined! error)

check with say $x = 2.000001$

$\lim_{x \to 2^+} (\ln(x-2)) = -\infty$

$\Longrightarrow \boxed{x = 2}$ is a vertical asymptote



Ex: Find the vertical asymptote, if any

$$f(x) = \frac{e^{-2x}}{x-5}$$

ANS Domain: $(-\infty, 5) \cup (5, +\infty)$

find a such that $f(a) =$ undefined!

$\Longrightarrow x-5=0 \longrightarrow x=5$, $a=5$

$$\lim_{x\to 5^+} f(x) = \longrightarrow \frac{\simeq e^{-10}}{0^+} \longrightarrow \frac{e^{-10}}{10^{-5}} \longrightarrow +\infty$$

$$\lim_{x\to 5^-} f(x) = \longrightarrow \frac{\simeq e^{-10}}{0^-} \longrightarrow \frac{e^{-10}}{-10^{-5}} \longrightarrow -\infty$$

Conclusion: $\boxed{x=5}$ vertical asymptote!

Ex: Find a such that the function is continuous on the real line

$$f(x) = \begin{cases} 4x^2 & x \geq 1 \\ ax-8 & x < 1 \end{cases}$$

Ans:

Recall: f is continuous at "c" if

$$\boxed{\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

So $f(1) = 4 = a(1) - 8 = \lim_{x\to 1^-} f(x) \implies a - 8 = 4$

$$\implies \boxed{a = 12}$$

Ex: $$f(x) = \begin{cases} 5 & x \leq -2 \\ ax+b & -2 < x < 3 \\ -5 & x \geq 3 \end{cases}$$

find the values of a and b such that f is continuous at -2 and 3

Ans: $f(-2) = 5$; $f(3) = -5$

$$\lim_{x\to -2^+} f(x) = a(-2) + b$$

$$\lim_{x\to 3^-} f(x) = a(3) + b$$

Continuity

$$\begin{cases} -2a+b = 5 \\ 3a+b = -5 \end{cases}$$

Elimination technique gives $\boxed{a = -2} \implies \boxed{b = 1}$

Recall case 2: $\frac{k}{0} \longrightarrow \infty$ (Note: $\frac{0}{k} = 0$, $k \neq 0$)

$\lim_{x \to 1^+} \left(\frac{x}{x^2 - 1} \right) \longrightarrow \frac{+1}{+0} \longrightarrow \boxed{+\infty}$

$\lim_{x \to 1^-} \left(\frac{x}{x^2 - 1} \right) \longrightarrow \frac{+1}{-0} \longrightarrow \boxed{-\infty}$

Note: $\lim_{x \to 1} \left(\frac{x}{x^2-1} \right) =$ DNE! b/c

Indeterminate form: $\frac{0}{0} \longrightarrow$ factorization to remove the "indetermination"

Ex: Evaluate

$$\lim_{x \to -1} \left(\frac{x+1}{x^2-1} \right) \longrightarrow \frac{0}{0}$$

Algebraically!

Note: $\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}, \quad x \neq -1$

So $\lim_{x \to -1} \left(\frac{x+1}{x^2-1} \right) = \lim_{x \to -1} \left(\frac{1}{x-1} \right) = \frac{+1}{-1-1} = \frac{+1}{-2} = \boxed{-\frac{1}{2}}$

Ex. Find $\lim_{x \to 3} \frac{x^3-27}{x-3}$ after you find a function $g(x)$ that agrees with the given function at all point but one

ANS

$$\lim_{x \to 3} \left(\frac{x^3-27}{x-3} \right)$$

Observe: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$x^3 - 27 = (x)^3 - (3)^3 = (x-3)(x^2+3x+9)$

So $\frac{x^3-27}{x-3} = \frac{(x-3)(x^2+3x+9)}{(x-3)} = x^2+3x+9 = g(x)$

$$\lim_{x \to 3} \left(\frac{x^3-27}{x-3} \right) = \lim_{x \to 3} \left(x^2+3x+9 \right) = \boxed{27}$$